Ecuatia: $x_{n+1} = ax_n + b$ (de ordinul I)

Solutia generala: $x_n = \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}$

Observatie: daca a=1 => $x_n = x_0 + nb$

Ecuatia: $x_{n+2} + ax_{n+1} + bx_n = 0$ (de ordinal II)

Daca r_1 si r_2 sunt radacinile ecuatiei: $r^2 + ar + b = 0$

Atunci solutia generala este:

$x_n = c_1 r_1^n + c_2 r_2^n$ daca r_1 diferit de r_2

$x_n = c_1 r_1^n + c_2 n r_1^n$ daca $r_1 = r_2$

Difference Equations: A linear first order difference equation with constant coefficients is:

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots$$

Here, a and b are constants. If $b = 0$, we obtain the homogeneous equation

$$x_{n+1} = ax_n, \quad n = 0, 1, 2, \ldots$$

The homogeneous equations are easy:

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2 x_0 \\ x_3 &= ax_2 = a^3 x_0 \\ x_4 &= ax_3 = a^4 x_0 \\ &\vdots \\ x_n &= ax_{n-1} = a^n x_0 \end{aligned}$$

In other words,

$$x_{n+1} = ax_n \quad \Rightarrow \quad x_n = a^n x_0$$

Let's try to solve the nonhomogeneous equation $x_{n+1} = ax_n + b$ in the same way:

$$\begin{aligned} x_1 &= ax_0 + b \\ x_2 &= ax_1 + b = a^2 x_0 + ab + b \\ x_3 &= ax_2 + b = a^3 x_0 + a^2 b + ab + b \\ &\vdots \\ x_n &= a^n x_0 + a^{n-1} b + \cdots + a^2 b + ab + b \end{aligned}$$

Consider the following identities:

$$\begin{aligned} (1-a)(1+a) &= (1-a^2) \\ (1-a)(1+a+a^2) &= (1-a^3) \\ (1-a)(1+a+a^2+a^3) &= (1-a^4) \\ &\vdots \end{aligned}$$

We can see that for the general case, the formula is:

$$(1-a)(1+a+a^2+a^3+\cdots+a^{n-1}) = (1-a^n)$$

We can easily show this by multiplying the expressions on the left and simplifying. If we rearrange this formula, we obtain:

$$1+a+a^2+a^3+\cdots+a^{n-1} = \frac{1-a^n}{1-a} = \frac{a^n-1}{a-1}, \qquad a \neq 1$$

If $a = 1$ we obtain:

$$1+a+a^2+a^3+\cdots+a^{n-1} = n$$

Now, we can express the solution of the nonhomogeneous difference equation

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots$$

as follows:

- If $a \neq 1$:

$$x_n = a^n x_0 + b\,\frac{a^n - 1}{a - 1}$$

- If $a = 1$:

$$x_n = x_0 + nb$$

Second Order Difference Equations: A second order linear homogeneous difference equation with constant coefficients is:

$$x_{n+2} + ax_{n+1} + x_n = 0, \quad n = 0, 1, 2, \ldots$$

where a and b are constants. We can try a solution of the form $x_n = r^n$. Then, we obtain:

$$r^{n+2} + ar^{n+1} + br^n = 0$$

$$r^2 + ar + b = 0$$

Note that, for this type of equation:

- A multiple of a solution is also a solution.
- Sum of two solutions is also a solution.

If there are two distinct roots r_1 and r_2, the general solution is:

$$x_n = c_1 r_1^n + c_2 r_2^n$$

If there is a double root r, the general solution is:

$$x_n = c_1 r^n + c_2 n r^n$$

(We do not consider complex roots in this course.)

If two initial conditions are given, we can determine c_1 and c_2. This method is very similar to the one we used for second order linear homogeneous differential equations.