Evolution of Complex Economic Systems and Uncertain Informations

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Abstract

Socio-economic networks describe collective phenomena through constraints relating actions of several agents, coalitions of these agents and multilinear connectionist operators acting on the set of actions of each coalition. We provide a class of control systems governing the evolution of actions, coalitions and multilinear connectionist operators under which the architecture of the network remains viable. The controls are the "viability multipliers" of the "resource space" in which the constraints are defined. They are involved as "tensor products" of the actions of the coalitions and the viability multiplier, allowing to encapsulate in this dynamical and multilinear framework the concept of Hebbian learning rules in neural networks in the form of "multi-Hebbian" dynamics in the evolution of connectionist operators. They are also involved in the evolution of coalitions through the "cost" of the constraints under the viability multiplier regarded as a price.

Introduction

We begin this paper by quoting the wish J. von Neumann and O. Morgenstern expressed in 1944 at the end of the first chapter of their monograph *"Theory of Games and Economic Behavior"*:

"Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore, preferable..."
"Our static theory specifies equilibria ... A dynamic theory, when one is found — will probably describe the changes in terms of simpler concepts."

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One of the economic characteristics is the presence of scarcity constraints, and more, generally, viability constraints to which a socio-economic system must *adapt* during its evolution.

It becomes then natural to specify the "minimal" conditions under which an economy can work and to specify classes — as large as possible — of reasonable economies whose evolution does not violate these viability conditions (as well as other specifications).

In my opinion, when one has to design a mathematical metaphor for an evolutionary model of socio-economic variables, one should start by gathering the constraints of these variables which cannot — or should not — be violated.

This requires first to delineate the endogenous states of the system under study and to discriminate them from the rest of the variables, regarded as exogenous, constituting in some sense the "environment" of the system under investigation. This partition between variables, which dictates the level of abstraction of a particular investigation, is the first source of constraints that the endogenous variable must obey.

Usually, there are few disputes among "modelers" when they are listing these constraints. Serious disagreements may begin when behavioral assumptions have to be made.

0.1 Designing Dynamics through Viability Multipliers

In order to weaken such controversies, or to prorogue the ultimate choice of a behavioral description of the economic agents, the strategy I suggest is to begin by characterizing an "envelope" of dynamical systems under which the constraints are viable, in the sense that starting from any initial state satisfying these constrains, at least one evolution is viable, i.e., satisfies at each instant these viability constraints. We shall perform this task here for a class of constraints describing the architecture of an abstract socio-economic network.

We then can devise general strategies for designing dynamical behaviors of the economic agents under which the constraints are viable. Now, the problem of choice of a behavior of the consumers is well circumscribed: one can propose, describe or suggest such and such shape of a change function and check whether or not a representative of this class belongs to this "envelope". Or one can propose a choice criterion and choose among viable dynamical economies of this "envelope" the ones which satisfy optimally this criterion.

Given the constraints that a socio-economic system must obey, and given an initial dynamic system for which these constraints are not viable, a theorem on viability multipliers allows us to correct the dynamics of the initial system in order that the constraints become viable under the corrected system. These viability multipliers play the role of Lagrange or Kuhn-Tucker multipliers in optimization theory, where an optimal solution of the problem of maximization of a utility function under constraint is obtained by maximizing without constraints a corrected utility function involving Lagrange multipliers. Both the viability and Lagrange multipliers belong to the same space (the dual of the resource space), and are usually interpreted as virtual prices as well as other regulatory controls, called regulons.

Therefore viability multipliers provide one way (but not the unique one) allowing to design dynamical economies for which the constrained set is viable, that should be familiar to economists, since it amounts to use for (virtual) prices the very same multipliers — viability multipliers instead of Lagrange ones — that are used in optimization under constraints for relaxing the constraints. In this respect, viability theory can be regarded as an evolution theory under (viability) constraints.

For example, ever since Adam Smith's invisible hand, what we call nowadays decentralization is justified by the need of agents to behave in a decentralized way for complying to scarcity constraints, using for this purpose "messages" such as prices or "rationing" mechanisms which involve shortages (and lines, queues, unemployment), or "frustration" of consumers, or "monetary" mechanisms, or others. "Prices" constitute the main examples of messages, actually, the messages with the smallest dimension (see for instance an introduction to this issue in [62, Saari]). Such prices appear here as viability multipliers emerging when allocations of commodities satisfy the scarcity constraints.

The next task is to derive from the confrontation of the (corrected) dynamics and the constraints the concealed regulation mechanisms governing viable evolutions, and to select some of them according to some further principle. This allows us to derive "adjustment laws" instead of founding the modelization process based on such a law. This goes against the tradition of theoretical economy, where the adjustment of some variables for reaching an equilibrium plays a basic and prominent role. The so–called "law of supply and demand" states that prices react in a determined direction in response to a difference between supply and demand in the market: the price of a particular commodity is assumed to vary according to the sign of the excess demand of this commodity. Instead of reasoning with a law of adjustment *a priori* given, and which does not produce viable evolutions, we shall build "dedicated" laws of supply and demand which shall provide viable solutions. In some way, this reverse approach allows us to "explain" *a posteriori* the role of such an adjustment law instead of scrutinizing the consequences of an *a priori* given law for possible justifications.

In summary, *the main purpose of the viability approach to dynamical economics is to explain the evolution of a system, governed by given nondeterministic dynamics and viability constraints, to reveal the concealed regulation laws which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

Therefore, using this viability approach, the modeling difficulty is confined to the elaboration of the list of constraints that the state of the system must obey and to use the above theory to suggest dynamics and study their properties.

0.2 Complex Economic Systems

This is at this level that the concept of "connectionist complexity" — to make more precise a meaning of such a polysemous concept as "complexity" — comes into the picture. The complexity of dynamic socio-economic systems stems from such a non teleological collective evolution of the set of agents, even though many individual economic agents think of themselves as being pursuing definitive and rational aims, instead of adapting permanently to the many viability constraints (among which scarcity constraints) they face under uncertainty, either contingent, stochastic or tychastic[1]. This theme has been introduced and studied in economic theory under the name of "bounded rationality". Indeed economic agents are humans, not computers, are seldom rational, obey inertia principles, are poor forecasters, actually very myopic. They base many decisions not on rational grounds, but on faith, beliefs and bets, rules of thumb, moods, rumors, and the like. They are more inductive than deductive in their learning processes. Actually, we can adopt Peirce's terminology and look at them as "abductive", i.e., as making conjectures rather than predictions[2]. They are most often more conservative than innovative, afraid of changes when they are not perceived as improving their situation. They may prefer to adopt a herd — or panurgean — behavior instead of choosing dissident ways opening new avenues.

Social (and living) systems are "complex", although there is no consensus on the definition of complexity. Reading the literature on complexity, and quoting George Cowan, the founder of the Santa Fe Institute, "in the universe, everything is connected with every thing" seems to be the consensual agreement of the members of this Institute. However, Seth Loyd had found 31 different definitions of complexity at the beginning of the 90's. This number increased a lot since. Complexity is indeed a polysemous word, that tries to embrace too many distinct phenomenon of interest in social and biological sciences[3].

[1] The theory of tychastic control (or "robust control") can be studied in the framework of dynamical games, when one player plays the role of Nature that chooses — plays — perturbations. These perturbations, disturbances, parameters that are not under the control of the controller or the decision-maker, could be called "random variables" if this vocabulary was not already confiscated by probabilists. We suggest to borrow to Charles Peirce the concept of *tyche*, one of the three words of classical Greek meaning "chance", and to call in this case the control system as a tychastic system. See [22, Aubin, Pujal & Saint-Pierre]) for more details.

[2] Non mathematical accounts of such questions can be found in [15, Aubin].

[3] Physicists and computer scientists have attempted to *measure* it in various ways, through the concept of Clausius's entropy, Claude Shannon's information, Gilbert Chauvet's nonsymmetric information, the degree of regularity instead of randomness, "hierarchical complexity" in the display of level of interactions, Andrei Kolmogorov, Gregory Chaitin & Ray Solomonoff "algorithmic in-

Since at least the works [42, Elton] of Charles Elton and [47, Hutchinson] of George Hutchinson at the end of the fifties, the conventional wisdom of biologists proposes in some loose ways that complexity, regarded as the number of variables of the systems and their links — is justified for maintaining "stability" — a fuzzy word, meaning confinement, or rather, viability as it was proposed to single out this meaning from the numerous intendments of "stability" in mathematics. Biodiversity is presently and actually championed on the basis of this objective. In a series of papers summarized in [52, May], Robert May and his collaborators disclaimed this proposition by showing that the higher the dimension, the less stable were dynamical models of Lotka-Volterra types. Therefore, either the biologists' assumption was false, or such mathematical connotation of complexity — the dimension of the state space — or the chosen mathematical model is inadequate. This is a suggestion made by John-Maynard Smith in [67, Smith], when he concluded that stability of ecosystems is due to some specific interactions.

Here, we retain the following features:

1. complexity means in the day-to-day language not only the number of variables of a system, but also and above all the labyrinth of connections between the components of an organization or a "system" (or, for that matter, of a living organism),
2. the purpose of complexity to sustain the "stability" — another polysemous is word — or more precisely the viability constraints set by an organization,
3. the increase of complexity is parallel to the growth of the web of constraints whenever the system cannot comply to them in an autonomous or decentralized way,
4. the organization of organisms as a hierarchical structure of relatively "autonomous" organs is due to "cycles" involved in the viability constraints or multi-stage production processes,
5. the organization in organisms or subsystems is rooted in the need to offer them slowly evolving partial environments to specialize them in specific activities.

formation contents" (see [34, Chaitin] for instance) and other temporal or spatial computational complexity indices measuring the computer time or the amount of computer memory needed to describe a system, "grammatical complexity" measuring the language to describe it, etc. Some economists link complexity issues with chaos theory as in [39, 40, Day] for instance. Other investigators link complexity issues with catastrophe theory, or fractals. See among many references [58, Peliti & Vulpiani].

Physicists — and among them, specialists of "spin glasses" such as Giorgio Parisi — propose the number of equilibria of a dynamical system as a characteristic of complexity. Or, even more to the point, "quasi equilibria", that are "small" areas of the state space in which the evolution remains a "long time", before "jumping quickly" to another quasi equilibrium (see [55, 56, 57, Parisi]).

The concept of *static and dynamical "connectionist complexity" indices* when connectionist matrices are used as regulons to regulate viable solutions and to the search of evolution minimizing at each instant those indices was introduced in [11, Aubin].

However, these attempts did not answer directly the question that some economists or biologists asked: *Complexity and hierarchical organization, yes, but for what purpose*?

This growth of "structural" complexity is the legacy that Jean-Baptiste de Monet, chevalier de Lamarck, offered to us, the backbone of his theory of evolution which was forgotten ever since, overshadowed as it was by other aspects of the evolution of species, such as the Darwinian natural selection or genetics. The Claude Bernard's "constance du milieu intérieur", the "homeostasis' of Walter Cannon, viability constraints to which dynamical systems must comply, later contributed to single out the crucial role of constraints as a key for explaining this aspect of complexity.

In this framework of adaptation to viability constraints, the evolution of the state no longer derives from intrinsic dynamical laws valid in the absence of constraints, but from some "organization" that evolves together with the state of the system in order to adapt to the viability constraints. This attempt to sustain the viability of the system by connecting the dynamics or the constraints of its agents may be a general feature of "complex systems".

We regard here *connectionism* — a less normative and more neutral term than *cooperation* whenever the system, the organ, the organism or the organization arise in economics, social sciences or biology— as an answer to adapt to more and more viability constraints, which implies the emergence of links between the components of a dynamical system and their evolution.

0.3 Connectionist Complexity of the Architecture of Networks

We shall restrict our study to the case when the organization is described by a "network" we now define.

A purpose of an organization is to the coordinate the actions of a finite number n of agents labelled $i = 1, \ldots, n$. It is descried here by the architecture of a network of agents, such as

1. socio-economic networks (see for instance [48, Ioannides], [11, 9, Aubin], [19, Aubin & Foray], [28, 27, Bonneuil].
2. neural networks (see for instance [8, 7, 12, Aubin]),
3. genetic networks (see for instance [**?**, 30, Bonneuil], [32, Bonneuil & Saint-Pierre])).

This coordinated activity requires a network of communications of actions $x_i \in X_i$ ranging over n finite dimensional vector spaces X_i.

The simplest general form of coordination is to require that a relation between actions of the form $g(A(x_1, \ldots, x_n)) \in M$ must be satisfied. Here

1. $A : \prod_{i=1}^n X_i \mapsto Y$ is a connectionist operator relating the individual actions in a collective way,

2. $M \subset Y$ is the subset of the resource space Y and g is a map, regarded as a resource map.

We shall study this coordination problem in a dynamic environment, by allowing actions $x(t)$ and connectionist operators $A(t)$ to evolve[4] according to dynamical systems we shall construct later. In this case, the coordination problem takes the form

$$\forall\, t \geq 0,\;\; g(A(t)(x_1(t), \ldots, x_n(t))) \;\in\; M$$

However, in the fields of motivation under investigation, the number n of variables may be very large. Even though the connectionist operators $A(t)$ defining the "architecture" of the network are allowed to operate *a priori* on all variables $x_i(t)$, they actually operate at each instant t on a coalition $S(t) \subset N := \{1, \ldots, n\}$ of such variables, varying naturally with time according to the nature of the coordination problem (see [14, Aubin], [59, Petrosjan], [60, Petrosjan & Zenkevitch] and [43, Filar & Petrosjan]) for closely related issues in dynamic cooperative game theory).

Therefore, our coordination problem in a dynamic environment involves the evolution

1. of actions $x(t) := (x_1(t), \ldots, x_n(t)) \in \prod_{i=1}^n X_i$,

2. of connectionist operators $A_{S(t)}(t) : \prod_{i=1}^n X_i \mapsto Y$,

3. acting on coalitions $S(t) \subset N := \{1, \ldots, n\}$ of the n agents

and requires that

$$\forall\, t \geq 0,\;\; g\left(\{A_S(t)(x(t))\}_{S \subset N}\right) \;\in\; M$$

where $g : \prod_{S \subset N} Y_S \mapsto Y$.

The question we raise is the following: Assume that we may know the intrinsic laws of evolution of the variables x_i (independently of the constraints), of the connectionist operator $A_S(t)$ and of the coalitions $S(t)$, there is no reason why collective constraints defining the above architecture are viable under these dynamics, i.e, satisfied at each instant.

One may be able, with a lot of ingeniosity and the intimate knowledge of a given problem, and for "simple constraints", to derive dynamics under which the constraints are viable.

[4]For simplicity, the set $M(t)$ is assumed to be constant. But they could also evolve through *mutational equations* and the following results can be adapted to this case. Curiously, the overall architecture is not changed when the set of available resources evolves under a mutational equation. See [13, Aubin] for more details on mutational equations.

However, we can use the kind of "mathematical factory" providing classes of dynamics "correcting" the initial (intrinsic) ones through viability multipliers $q(t)$ ranging over the dual Y^* of the resource space Y in such a way that the viability of the constraints is guaranteed.

This may allow us to provide an explanation of the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves.

The results presented here use this approach in the case of the above specific constraints. We show that by doing so, the dynamics of the evolution of connectionist operators and coalitions present some interesting features.

In order to tackle mathematically this problem, we shall

1. restrict the connectionist operators to be multiaffine, the sum over all coalitions of multilinear operators A_S, also called (or regarded) as tensors[5], and thus, involve tensor products,

2. next, allow coalitions S to become fuzzy coalitions so that they can evolve continuously.

Fuzzy coalitions $\chi = (\chi_1, \ldots, \chi_n)$ are defined by memberships $\chi_i \in [0, 1]$ between 0 and 1, instead of being equal to either 0 or 1 as in the case of usual coalitions. The membership $\gamma_S(\chi) := \prod_{i \in S} \chi_i$ is by definition the product of the memberships of the members $i \in S$ of the coalitions. Using fuzzy coalitions allows us to define their velocities and study their evolution.

The viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order that the viability constraints describing the network can be satisfied at each instant, and the main theorem of this paper guarantees that it is possible. Another one tells us how to choose at each instant such regulons (the regulation law).

The main theorem asserts that for each agent i, the velocities $x_i'(t)$ of the state and the velocities $\chi_i'(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by adding

1. the sum over all coalitions S to which he belongs of adequate functions weighted by the membership $\gamma_S(\chi(t))$,

2. the sum over all coalitions S to which he belongs of the costs of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S \setminus i}(\chi(t))$. This type of dynamics describes a *panurgean*

[5]that are nothing other than matrices when the operators are linear instead of multilinear. Tensors are the matrices of multilinear operators, so to speak, and their "entries" depend upon several indexes instead of the two involved in matrices.

effect. The (algebraic) increase of agent i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the agents of the coalition other than him.

As for the correction of the velocities of the connectionist tensors A_S, their correction is a weighted "multi-Hebbian" rule: for each component of the connectionist tensor, the correction term is the product of the membership $\gamma S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon. This is a generalization of the celebrated Hebbian rule proposed by Hebb in his classic book *The organization of behavior* in 1949 as the basic learning process of synaptic weight in neural networks (see [8, 7, 12, Aubin]) for more details). Mathematically speaking, we recognize tensor products of vectors that boil down to matrices when only two vectors are involved.

In other words, the viability multipliers appear in the regulation of the multiaffine connectionist operators under the form of a "multi-Hebbian" rules, as in [16, Aubin & Burnod] where they were introduced for the fist time, compounded with the presence of the membership coalitions $\gamma_S(\chi(t))$ when coalitions of agents are allowed to form and to evolve.

Even though viability multipliers do not provide all the dynamics under which a constrained set is viable, they provide classes of them exhibiting interesting structures that deserve to be investigated and tested in concrete situations.

Remark: Learning Laws and Supply and Demand Law — It is curious that both the standard supply and demand law, known as the Walrasian tâtonnement process, in economics and the Hebbian learning law in cognitive sciences were the starting points of the Walras general equilibrium theory and of learning processes in neural networks. In both theories, this choice of putting such adaptation laws as a prerequisite led to the same cul de sacs. As we alluded to above, starting instead from dynamic laws of agents, viability theory provides "dedicated adaptation laws", so to speak, as the conclusion of the theory instead as the primitive feature. In both cases, the point is to maintain the viability of the system, that allocation of scarce commodities satisfy the scarcity constraints in economics, that the viability of the neural network is maintained in the cognitive sciences. For neural networks, this approach provides learning rules that possess the features meeting the Hebbian criterion. For the general networks studied here, these features are still satisfied in spirit. □

These modeling challenges raised by the study of the evolution of socio-economic networks require not necessarily more difficult mathematical techniques, but new ones motivated by these questions. If we accept that physics studies much simpler phenomena than the ones investigated by social and biological sciences, and that for this very purpose, they motivated and used a more and more complex mathematical apparatus, we have to accept also that social sciences require a new and dedicated mathematical arsenal which goes beyond what is presently available. Paradoxically, the very fact that the mathematical tools useful for social sciences are and have to be quite sophisticated impairs their acceptance by many social scientists and economists, and the gap menaces to widen.

0.4 Outline

We present examples of network structures in order of increasing difficulty. We begin with results (already) obtained for affine constraints (case of one agent), and expose them in details when there are only two agents and when bilinear constraints are involved.

In the next section, we exhibit the results for n agents for multiaffine constraints without evolving coalitions, whereas in the last section, we introduce fuzzy coalitions and show how they may evolve for maintaining the viability of the architecture of the network.

1 Examples of Architectures Involving Linear and Bilinear Connectionist Maps

1.1 Case of Affine Constraints

For simplicity, we summarize the case when there is only one agent and when the operator $A : X \mapsto Y$ is affine studied in [12, 9, 11, Aubin]:

$$\forall\, x \in X,\;\; A(x) \;:=\; Wx + y \;\text{ where }\; W \in \mathcal{L}(X, Y) \;\&\; y \in Y$$

The coordination problem takes the form:

$$\forall\, t \geq 0,\;\; W(t)x(t) + y(t) \;\in\; M$$

where both the state x, the resource y and the connectionist operator W evolve. These constraints are not necessarily viable under an arbitrary dynamic system of the form

$$\begin{cases} i) & x'(t) \;=\; c(x(t)) \\ ii) & y'(t) \;=\; d(y(t)) \\ iii) & W'(t) \;=\; \alpha(W(t)) \end{cases}$$

We can reestablish viability by involving multipliers $q \in Y^\star$ ranging over the dual $Y^\star := Y$ of the resource space Y. We denote by $W^\star \in \mathcal{L}(Y^\star, X^\star)$ the transpose of W:

$$\forall\, q \in Y^\star,\; \forall\, x \in X,\;\; \langle W^\star q, x\rangle \;:=\; \langle q, Wx\rangle$$

by $x \otimes q \in \mathcal{L}(X, Y^*)$ the tensor product defined by

$$x \otimes q \ : \ p \in X^* := X \mapsto (x \otimes q)(p) \ := \ \langle p, x \rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$.

The contingent cone $T_M(x)$ to $M \subset Y$ at $y \in M$ is the set of directions $v \in Y$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $y + h_n v_n \in M$ for every n. The (regular) normal cone to $M \subset Y$ at $y \in M$ is defined by

$$N_M(y) := \{q \in Y^* \mid \forall\, v \in T_M(y),\ \langle q, v \rangle \leq 0\}$$

(see [20, Aubin & Frankowska] and [61, Rockafellar & Wets] for more details on these topics).

We can prove that the viability of the constraints can be reestablished when the initial system is replaced the above system by the control system

$$\begin{cases} i) & x'(t) \ = \ c(x(t)) - W^*(t)q(t) \\ ii) & y'(t) \ = \ d(y(t)) - q(t) \\ iii) & W'(t) \ = \ \alpha(W(t)) - x(t) \otimes q(t) \\ & \text{where } q(t) \ \in \ N_M(W(t)x(t) + y(t)) \end{cases}$$

where $N_M(y) \subset Y^*$ denotes the normal cone to M at $y \in M \subset Y$ and $x \otimes q \in \mathcal{L}(X, Y^*)$ denotes the tensor product defined by

$$x \otimes q \ : \ p \in X^* := X \mapsto (x \otimes q)(x) \ := \ \langle p, x \rangle q$$

the matrix of which is made of entries $(x \otimes q)_i^j = x_i q^j$ (see [8, Aubin] for more details on the relations between Hebbian rules and tensor products in the framework of neural networks).

In other words, the correction of a dynamical system for reestablishing the viability of constraints of the form $W(t)x(t) + y(t) \in M$ involves the celebrated Hebbian rule proposed by Hebb in 1949 as the basic learning process of synaptic weight: Taking $\alpha(W) = 0$, the evolution of the synaptic matrix $W := (w_i^j)$ obeys the differential equation

$$\frac{d}{dt} w_i^j(t) \ = \ -x_i(t) q^j(t)$$

It states that the velocity of the synaptic weight is the product of the presynaptic activity and the postsynaptic activity. This intuition of a neurobiologist is confirmed mathematically by the above result. Such a learning rule "pops up" (or, more pedantically, emerges) whenever the synaptic matrices are involved for regulating the system in order to maintain the "homeostatic" constraint $W(t)x(t) + y(t) \in M$.

We may enrich this problem by introduced a coefficient $\chi(t) \in [0,1]$ aimed at "tuning" the action $x(t)$ regarded as a potential action that is not wholly implemented. In this framework, the constraint becomes

$$\forall\, t \geq 0,\;\; W(t)\chi(t)x(t) + y(t) \;\in\; M$$

Again, one can correct a differential system of the form

$$\begin{cases} i) & x'(t) \;=\; c(x(t)) \\ ii) & y'(t) \;=\; d(y(t)) \\ iii) & \chi'(t) \;=\; \kappa(\chi(t)) \\ iv) & W'(t) \;=\; \alpha(W(t)) \end{cases}$$

by introducing viability multipliers as controls in a system of the form

$$\begin{cases} i) & x'(t) \;=\; c(x(t)) - W^*(t)q(t) \\ ii) & y'(t) \;=\; d(y(t)) - q(t) \\ iii) & \chi'(t) \;=\; \kappa(\chi(t)) - \langle q(t), W(t)x(t)\rangle \\ iv) & W'(t) \;=\; \alpha(W(t)) - x(t) \otimes q(t) \\ & \text{where } q(t) \;\in\; N_M(W(t)\chi(t)x(t) + y(t)) \end{cases}$$

The correction term is the "cost of the linear constraint" $\langle q(t), W(t)x(t)\rangle$ in the law of evolution of $\chi(t)$.

1.2 Case of Bi-Affine Constraints

Before investigating the general case and confronting notational difficulties, let us explain how we go from the affine case to the bi-affine case.

Here, we assume now that $X := X_1 \times X_2$ is the product of two vector spaces. Affine constraints takes the form

$$\forall\, t \geq 0,\;\; A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t) \;\in\; M$$

where $A_i \in \mathcal{L}(X_i, Y)$ $(i = 1, 2)$ and $A_\emptyset \in Y$. But we can also involve a bilinear operator $A_{\{1,2\}} \in \mathcal{L}_2(X_1 \times X_2, Y)$ and consider bi-affine constraints of the form:

$$\forall\, t \geq 0,\;\; A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t) \;\in\; M$$

We introduce the linear operators $A_{\{1,2\}}(x_i) \in \mathcal{L}(X_{-i}, Y)$ defined by

$$A_{\{1,2\}}(x_1) : x_2 \mapsto A_{\{1,2\}}(x_1)x_2 := A_{\{1,2\}}(x_1, x_2)$$

and

$$A_{\{1,2\}}(x_2) : x_1 \mapsto A_{\{1,2\}}(x_2)x_1 := A_{\{1,2\}}(x_1, x_2)$$

We shall prove that when these constraints are not viable under an arbitrary dynamic system of the form

$$\left\{\begin{array}{lrcl} i) & x_i'(t) & = & c_i(x(t)),\ i=1,2 \\ ii) & A_\emptyset'(t) & = & \alpha_\emptyset(A_\emptyset(t)) \\ iii) & A_1'(t) & = & \alpha_1(A_1(t)) \\ iv) & A_2'(t) & = & \alpha_2(A_2(t)) \\ v) & A_{\{1,2\}}'(t) & = & \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) \end{array}\right.$$

we can still reestablish viability by involving multipliers $q \in Y^*$ and correct the above system by the control system

$$\left\{\begin{array}{lrcl} i) & x_1'(t) & = & c_1(x(t)) - A_1(t)^*q(t) - A_{\{1,2\}}(t)(x_2(t))^*q(t) \\ ii) & x_2'(t) & = & c_2(x(t)) - A_2(t)^*q(t) - A_{\{1,2\}}(t)(x_1(t))^*q(t) \\ iii) & A_\emptyset'(t) & = & \alpha_\emptyset(A_\emptyset(t)) - q(t) \\ iv) & A_1'(t) & = & \alpha_1(A_1(t)) - x_1(t)\otimes q(t) \\ v) & A_2'(t) & = & \alpha_2(A_2(t)) - x_2(t)\otimes q(t) \\ vi) & A_{\{1,2\}}'(t) & = & \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) - x_1(t)\otimes x_2(t)\otimes q(t) \\ & \multicolumn{3}{l}{\text{where } q(t) \in N_M(A_{\{1,2\}}(t)(x_1(t),x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) + A_\emptyset(t))} \end{array}\right.$$

Hence, the structure of this control system involves the transposes $A_i^*(t)q(t)$ and $A_{\{1,2\}}(t)(x_j(t))^*(t)q(t)$ $(i = 1,2)$ in the evolution of the variables $x_i(t)$, and the tensor products $x_i(t)\otimes q(t)$ (Hebbian rules) in the evolution of the linear operators $A_i(t)$, and the tensor product $x_1(t)\otimes x_2(t)\otimes q(t)$ in the evolution of the bilinear form $A_{\{1,2\}}$.

The tensor product $x_1 \otimes x_2 \otimes q$ is a bilinear operator from $X_1^* \times X_2^*$ to Y^* associating with any pair $(p_1, p_2) \in X_1^* \times X_2^*$ the element

$$(x_1 \otimes x_2 \otimes q)(p_1,p_2) \ := \ \langle p_1, x_1\rangle\langle p_2, x_2\rangle q$$

If the vector spaces are supplied with bases, the components of this bilinear form — the "tensors" — can be written

$$(x_1 \otimes x_2 \otimes q)^j_{i_1,i_2} \ = \ x_{1_{i_1}} x_{2_{i_2}} q^j$$

as the products of the components of the three fagents of this tensor product. Taking $\alpha_{1,2}(A) = 0$, the evolution of the bi-synaptic tensor $A_{\{1,2\}} := (a^j_{i_1,i_2})$ obeys the differential equation

$$\frac{d}{dt}a^j_{i_1,i_2}(t) \ = - \ x_{1_{i_1}}(t) x_{2_{i_2}}(t) q^j(t)$$

It states that the velocity of the synaptic tensor is the product of the presynaptic activities of the neurons arriving at the synapse (i_1, i_2, j) and the postsynaptic activity (see [16, Aubin & Burnod]).

We may enrich this problem by introduced coefficients $\chi_i(t) \in [0,1]$ aimed at tuning the action $x_i(t)$ $(i = 1,2)$ that we shall regard as the components of a fuzzy coalition later. In this framework, the constraint becomes: $\forall\, t \geq 0$,

$$\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t),x_2(t)) + \chi_1(t)A_1(t)x_1(t) + \chi_2(t)A_2(t)x_2(t) + A_\emptyset(t) \ \in \ M$$

If we assume that the evolutions of these $\chi_i(t)$ are governed by differential equations

$$\chi_i'(t) \ = \ \kappa_i(\chi_i(t)), \ \ i = 1,2$$

we shall prove that the above constraints are viable under the control system

$$\left\{\begin{array}{lrcl}
i) & x_1'(t) & = & c_1(x(t)) - \chi_1(t)A_1(t)^\star q(t) - \chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_2(t))^\star q(t) \\
ii) & x_2'(t) & = & c_2(x(t)) - \chi_2(t)A_2(t)^\star q(t) - \chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t))^\star q(t) \\
iii) & \chi_1'(t) & = & \kappa_1(\chi_1(t)) - \langle q(t), A_1(t)x_1(t) + \chi_2(t)A_{\{1,2\}}(t)(x_1(t),x_2(t))\rangle \\
iv) & \chi_2'(t) & = & \kappa_2(\chi_2(t)) - \langle q(t), A_2(t)x_2(t) + \chi_1(t)A_{\{1,2\}}(t)(x_1(t),x_2(t))\rangle \\
v) & A_\emptyset'(t) & = & \alpha_\emptyset(A_\emptyset(t)) - q(t) \\
vi) & A_1'(t) & = & \alpha_1(A_1(t)) - \chi_1(t)x_1(t) \otimes q(t) \\
vii) & A_2'(t) & = & \alpha_2(A_2(t)) - \chi_2(t)x_2(t) \otimes q(t) \\
viii) & \multicolumn{3}{l}{A_{\{1,2\}}'(t) \ = \ \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) - \chi_1(t)\chi_2(t)x_1(t)\otimes x_2(t)\otimes q(t) \text{ where}} \\
 & \multicolumn{3}{l}{q(t) \ \in \ N_M(\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t),x_2(t)) + \chi_1(t)A_1(t)x_1(t)} \\
 & \multicolumn{3}{l}{+\chi_2(t)A_2(t)x_2(t) + A_\emptyset(t))}
\end{array}\right.$$

2 Regulation by Connectionist Tensors

2.1 Connectionist Tensors

In order to handle more explicit and tractable formulas and results, we shall assume that the connectionist operator $A : X := \prod_{i=1}^n X_i \rightsquigarrow Y$ is multiaffine.

For defining such a multiaffine operator, we associate with any coalition $S \subset N$ its characteristic function $\chi_S : N \mapsto \mathbf{R}$ associating with any $i \in N$

$$\chi_S(i) \ := \left\{\begin{array}{ll} 1 & \text{if} \ \ i \in S \\ 0 & \text{if} \ \ i \notin S \end{array}\right.$$

It defines a linear operator $\chi_S \circ \ \in \ \mathcal{L}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n X_i\right)$ that associates with any $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ the sequence $\chi_S \circ x \in \mathbf{R}^n$ defined by

$$\forall \ = 1, \ldots, n, \ \ (\chi_S \circ x)_i \ := \left\{\begin{array}{ll} x_i & \text{if} \ \ i \in S \\ 0 & \text{if} \ \ i \notin S \end{array}\right.$$

We associate with any coalition $S \subset N$ the subspace

$$X^S \ := \ x_S \circ \prod_{i=1}^n X_i \ = \ \left\{ x \in \prod_{i=1}^n X_i \ \ \text{such that} \ \forall\, i \notin S, \ \ x_i = 0 \right\}$$

since $x_S\circ$ is nothing other that the canonical projector from $\prod_{i=1}^n X_i$ onto X^S. In particular, $X^N := \prod_{i=1}^n X_i$ and $X^\emptyset := \{0\}$.

Let Y be another finite dimensional vector space. We associate with any coalition $S \subset N$ the space $\mathcal{L}_S(X^S, Y)$ of S-linear operators A_S. We extend such a S-linear operator A_S to a n-linear operator (again denoted by) $A_S \in \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ defined by:

$$\forall\, x \in \prod_{i=1}^n X_i, \;\; A_S(x) \;=\; A_S(x_1, \ldots, x_n) \;:=\; A_S(\chi_S \circ x)$$

A multiaffine operator $A \in \mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$ is a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions:

$$A(x_1, \ldots, x_n) \;:=\; \sum_{S \subset N} A_S(\chi_S \circ x) \;=\; \sum_{S \subset N} A_S(x)$$

We identify $A_\emptyset$ with a constant $A_\emptyset \in Y$.

Hence the collective constraint linking multiaffine operators and actions can be written in the form

$$\forall\, t \geq 0, \;\; \sum_{S \subset N} A_S(t)(x(t)) \;\in\; M$$

For any $i \in S$, we shall denote by $(x_{-i}, u_i) \in X^N$ the sequence $y \in X^N$ where $y_j := x_j$ when $j \neq i$ and $y_i = u_i$ when $j = i$.

We shall denote by $A_S\,(x_{-i}) \in \mathcal{L}(X_i, Y)$ the linear operator defined by $u_i \mapsto A_S(x_{-i})u_i := A_S(x_{-i}, u_i)$. We shall use its transpose $A_S\,(x_{-i})^\star \in \mathcal{L}(Y^\star, X_i^\star)$ defined by

$$\forall\, q \in Y^\star, \; \forall\, u_i \in X_i, \;\; \langle A_S\,(x_{-i})^\star\, q, u_i \rangle \;=\; \langle q, A_S\,(x_{-i})\, u_i \rangle$$

We associate with $q \in Y^\star$ and elements $x_i \in X_i$ the multilinear operator[6]

$$x_1 \otimes \cdots \otimes x_n \otimes q \;\in\; \mathcal{L}_n\left(\prod_{i=1}^n X_i^\star, Y^\star\right)$$

associating with any $p := (p_1, \ldots, p_n) \in \prod_{i=1}^n X_i^\star$ the element $\left(\prod_{i=1}^n \langle p_i, x_i \rangle\right)\; q$:

$$x_1 \otimes \cdots \otimes x_n \otimes q : p := (p_1, \ldots, p_n) \in \prod_{i=1}^n X_i^\star \mapsto (x_1 \otimes \cdots \otimes x_n \otimes q)(p) := \left(\prod_{i=1}^n \langle p_i, x_i \rangle\right) q \in Y^\star$$

[6] We recall that the space $\mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ of n-linear operators from $\prod_{i=1}^n X_i$ to Y is isometric to the tensor product $\bigotimes_{i=1}^n X_i^\star \otimes Y$, the dual of which is $\bigotimes_{i=1}^n X_i \otimes Y^\star$, that is isometric with $\mathcal{L}_n\left(\prod_{i=1}^n X_i^\star, Y^\star\right)$.

This multilinear operator $x_1 \otimes \cdots \otimes x_n \otimes q$ is called the tensor product of the x_i's and q.

We recall that the duality product on $\mathcal{L}_n\left(\prod_{i=1}^n X_i^*, Y^*\right) \times \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ for pairs $(x_1 \otimes \cdots \otimes x_n \otimes q, A)$ can be written in the form:

$$\langle x_1 \otimes \cdots \otimes x_n \otimes q, A\rangle \ := \ \langle q, A(x_1, \ldots, x_n)\rangle$$

2.2 Multi-Hebbian Learning Process

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\begin{cases} i) & x_i'(t) \ = \ c_i(x(t)), \ i = 1, \ldots, n \\ ii) & A_S'(t) \ = \ \alpha_S(A(t)), \ S \subset N \end{cases}$$

Using viability multipliers, we can modify the above dynamics by introducing regulees that are elements $q \in Y^*$ of the dual Y^* of the space Y:

Theorem 2.1 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall\, t \geq 0, \ \sum_{S \subset N} A_S(t)(x(t)) \ \in \ M$$

are viable under the control system

$$\begin{cases} i) & x_i'(t) \ = \ c_i(x_i(t)) - \displaystyle\sum_{S \ni i} A_S(t)(x_{-i}(t))^* q(t), \ i = 1, \ldots, n \\ & \\ ii) & A_S'(t) \ = \ \alpha_S(A(t)) - \left(\displaystyle\bigotimes_{j \in S} x_j(t)\right) \otimes q(t), \ S \subset N \\ & \text{where } q(t) \in N_M\left(\sum_{S \subset N} A_S(t)(x(t))\right) \end{cases}$$

Remark: Multi-Hebbian Rule — When we regard the multilinear operator A_S as a tensor product of components $A^j_{S_{\Pi_{i \in S} i_k}}$, $j = 1, \ldots, p$, $i_k = 1, \ldots, n_i$, $i \in S$, differential equation ii) can be written in the form: $\forall\ i \in S,\ j = 1, \ldots, p,\ k = 1, \ldots, n_i$,

$$\frac{d}{dt} A^j_{S_{\Pi_{i \in S} i_k}} \ = \ \alpha_{S_{\Pi_{i \in S} i_k}}(A(t)) - \left(\prod_{i \in S} x_{i_k}(t)\right) q^j(t)$$

The correction term of the component $A^j_{S_{\Pi_{i \in S} i_k}}$ of the S-linear operator is the product of the components $x_{i_k}(t)$ actions x_i in the coalition S and of the component q^j of the viability multiplier. *This can be regarded as a multi-Hebbian rule in neural* *network learning algorithms*, since for linear operators, we find the product of the component x_k of the pre-synaptic action and the component q^j of the post-synaptic action. $\square$

Indeed, when the vector spaces $X_i := \mathbf{R}^{n_i}$ are supplied with basis e^{i_k}, $k = 1, \ldots, n_i$, when we denote by $e^*_{i_k}$ their dual basis, and when $Y := \mathbf{R}^p$ is supplied with a basis f^j, its dual supplied with the dual basis f^*_j, then the tensor products $\left(\bigotimes_{i \in S} e^{i_k}\right) \otimes f^*_j$ $(j = 1, \ldots, p,\ k = 1, \ldots, n_i)$ form a basis of $\mathcal{L}_S\left(X^{S^*}, Y^*\right)$.

Hence the components of the tensor product $\left(\bigotimes_{i \in S} x_i\right) \otimes q$ in this basis are the products $\left(\prod_{i \in S} x_{i_k}\right) q^j$ of the components q^j of q and x_{i_k} of the x_i's, where $q^j := \langle q, f^j \rangle$ and $x_{i_k} := \langle e^*_{i_k}, x_i \rangle$. Indeed, we can write

$$\left(\bigotimes_{i \in S} x_i\right) \otimes q \ = \ \sum_{j=1}^{p} \sum_{i \in S} \sum_{k=1}^{n_i} \left(\langle q, f^j \rangle \prod_{i \in S} \langle e^*_{i_k}, x_i \rangle\right) \left(\bigotimes_{i=1}^{n} e^{i_k}\right) \otimes f^*_j$$

3 Regulation Involving Fuzzy Coalitions

3.1 Fuzzy Coalitions

This first definition of a coalition which comes to mind being that of a subset of players $S \subset N$ is not adequate for tackling dynamical models of evolution of coalitions since the 2^n coalitions range over a finite set, preventing us from using analytical techniques.

One way to overcome this difficulty is to embed the family of subsets of a (discrete) set N of n players to the space $\mathbf{R}^n$ through the map χ associating with any coalition $S \in \mathcal{P}(N)$ its characteristic function[7] $\chi_S \in \{0,1\}^n \subset \mathbf{R}^n$, since $\mathbf{R}^n$ can be regarded as the set of functions from N to $\mathbf{R}$.

By definition, the family of fuzzy sets[8] is the convex hull $[0,1]^n$ of the power set

[7] This canonical embedding is more adapted to the nature of the power set $\mathcal{P}(N)$ than the universal embedding of a discrete set M of m elements to $\mathbf{R}^m$ by the Dirac measure associating with any $j \in M$ the jth element of the canonical basis of $\mathbf{R}^m$. The convex hull of the image of M by this embedding is the probability simplex of $\mathbf{R}^m$. Hence fuzzy sets offer a "dedicated convexification" procedure of the discrete power set $M := \mathcal{P}(N)$ instead of the universal convexification procedure of frequencies, probabilities, mixed strategies derived from its embedding in $\mathbf{R}^m = \mathbf{R}^{2^n}$.

[8] This concept of fuzzy set was introduced in 1965 by L. A. Zadeh. Since then, it has been wildly successful, even in many areas outside mathematics!. Lately, we found in *"La lutte finale"*, Michel Lafon (1994), p.69 by A. Bercoff the following quotation of the late François Mitterand, president of the French Republic (1981-1995): *"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...* (Today, we swim in the pure poetry of fuzzy subsets)!

$\{0,1\}^n$ in $\mathbf{R}^n$. Therefore, we can write any fuzzy set in the form

$$\chi \;=\; \sum_{S\in\mathcal{P}(N)} m_S\chi_S \;\text{ where }\; m_S\geq 0 \;\;\&\;\; \sum_{S\in\mathcal{P}(N)} m_S = 1$$

The memberships are then equal to

$$\forall\, i\in N,\;\; \chi_i \;=\; \sum_{S\ni i} m_S$$

Consequently, if m_S is regarded as the probability for the set S to be formed, the membership of the player i to the fuzzy set[9] χ is the sum of the probabilities of the coalitions to which player i belongs. Player i participates fully in χ if $\chi_i = 1$, does not participate at all if $\chi_i = 0$ and participates in a fuzzy way if $\chi_i \in]0,1[$. We associate with a fuzzy coalition χ the set $P(\chi) := \{i\in N \mid \chi_i \neq 0\} \subset N$ of agents i partipating to the fuzzy coalition χ.

We also introduce the membership

$$\gamma_S(\chi) \;:=\; \prod_{j\in S}\chi_j$$

of a coalition S in the fuzzy coalition χ as the product of the memberships of agents i of the coalition S. It vanishes whenvever one the membership of one agent does and boils down to individual memberships for one agent coalitions. when two coalitions are disjoint ($S\cap T = \emptyset$), then $\gamma_{S\cup T}(\chi) = \gamma_S(\chi)\gamma_T(\chi)$. In particular, for any agent $i\in S$, $\gamma_S(\chi) = \chi_i\gamma_{S\setminus i}(\chi)$

Let $A\in\mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$, a sum of S-linear operators $A_S\in\mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions, be a multiaffine operator.

When χ is a fuzzy coalition, we observe that

$$A(\chi\circ x) \;=\; \sum_{S\subset P(\chi)}\gamma_S(\chi)A_S(x) \;=\; \sum_{S\subset P(\chi)}\left(\prod_{j\in S}\chi_j\right)A_S(x)$$

We wish to encapsulate the idea that at each instant, only a number of fuzzy coalitions χ are active. Hence the collective constraint linking multiaffine operators, fuzzy coalitions and actions can be written in the form

$$\forall\, t\geq 0,\;\; \sum_{S\subset P(\chi(t))}\gamma_S(\chi(t))A_S(t)(x(t)) \;=\; \sum_{S\subset P(\chi(t))}\left(\prod_{j\in S}\chi_j(t)\right)A_S(t)(x(t)) \;\in\; M$$

[9] Actually, this idea of using fuzzy coalitions has already been used in the framework of cooperative games with and without side-payments (see [2, 3, Aubin], [1, Aubin, Chapter 12] and [10, Aubin, Chapter 13],the books [51, Mares] and [54, Mishizaki & Sokawa], [24, 25, 26, Basile], [23, Basile, De Simone & Graziano], [44, Florenzano]). Fuzzy coalitions have also been used in dynamical models of cooperative games in [17, Aubin & Cellina, Chapter 4] and of economic theory in [9, Aubin, Chapter 5].

3.2 Constructing Viable Dynamics

Assume that we start with intrinsic dynamics of the actions x_i, the resources y, the connectionist matrices W and the fuzzy coalitions χ:

$$\begin{cases} i) & x_i'(t) \;=\; c_i(x(t)),\; i = 1, \ldots, n \\ ii) & \chi_i'(t) \;=\; \kappa_i(\chi(t)),\; i = 1, \ldots, n \\ iii) & A_S'(t) \;=\; \alpha_S(A(t)),\; S \subset N \end{cases}$$

Using viability multipliers, we can modify the above dynamics by introducing regulees that are elements $q \in Y^*$ of the dual Y^* of the space Y:

Theorem 3.1 *Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed. Then the constraints*

$$\forall\, t \geq 0, \quad \sum_{S \subset P(\chi(t))} A_S(t)(\chi(t) \circ x(t)) \;=\; \sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \;\in\; M$$

are viable under the control system

$$\begin{cases} i) & x_i'(t) \;=\; c_i(x_i(t)) - \displaystyle\sum_{S \ni i} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x_{-i}(t))^* q(t),\; i = 1, \ldots, n \\ \\ ii) & \chi_i'(t) \;=\; \kappa_i(\chi(t)) - \displaystyle\sum_{S \ni i} \left(\prod_{j \in S \setminus i} \chi_j(t) \right) \langle q(t), A_S(t)\,(x(t)) \rangle\,,\; i = 1, \ldots, n \\ \\ iii) & A_S'(t) \;=\; \alpha_S(A(t)) - \left(\displaystyle\prod_{j \in S} \chi_j(t) \right) \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q(t),\; S \subset N \\ & \text{where } q(t) \in N_M(\sum_{S \subset P(\chi(t))} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)))) \end{cases}$$

Let us comment these formulas. First, the viability multipliers $q(t) \in Y^*$ can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order that the viability constraints describing the network can be satisfied at each instant, and the above theorem guarantees that it is possible. The next section tells us how to choose at each instant such regulons (the regulation law).

For each agent i, the velocities $x_i'(t)$ of the state and the velocities $\chi_i'(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by subtracting

1. the sum over all coalitions S to which he belongs of the $A_S(t)(x_{-i}(t))^* q(t)$ weighted by the membership $\gamma_S(\chi(t))$:

$$x_i'(t) \;=\; c_i(x_i(t)) - \sum_{S \ni i} \gamma_S(\chi(t)) A_S(t)(x_{-i}(t))^* q(t)$$

2. the sum over all coalitions S to which he belongs of the costs $\langle q(t), A_S(t)\,(x(t))\rangle$ of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S\backslash i}(\chi(t))$:

$$\chi_i'(t) \;=\; \kappa_i(\chi(t)) - \sum_{S\ni i} \gamma_{S\backslash i}(\chi(t))\,\langle q(t), A_S(t)\,(x(t))\rangle$$

This type of dynamics describes a *panurgean effect*. The (algebraic) increase of agent i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the agents of the coalition other than him.

As for the correction of the velocities of the connectionist tensors A_S, their correction is a weighted "multi-Hebbian" rule: for each component $A^j_{S_{\Pi_{i\in S} i_k}}$ of A_S, the correction term is the product of the membership $\gamma S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon:

$$\frac{d}{dt}A^j_{S_{\Pi_{i\in S} i_k}} \;=\; \alpha_{S_{\Pi_{i\in S} i_k}}(A(t)) - \gamma_S(\chi(t))\left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)$$

3.3 The Regulation Map

Actually, the viability multipliers $q(t)$ regulating viable evolutions of the actions $x_i(t)$, the fuzzy coalitions $\chi(t)$ and the multiaffine operators $A(t)$ obey the regulation law (an "adjustment law", in the vocabulary of economists) of the form

$$\forall\, t \geq 0,\;\; q(t) \;\in\; R_M(x(t), \chi(t), A(t))$$

where $R_M : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y) \rightsquigarrow Y^\star$ is the regulation map R_M that we can compute.

For that purpose, we introduce the operator $h : X^N \times \mathbf{R}^n \times \mathcal{A}_n(X^N, Y)$ defined by

$$h(x, \chi, A) \;:=\; \sum_{S\subset N} A_S(\chi \circ x)$$

and the linear operator $H(x, \chi, A) : Y^\star := Y \mapsto Y$ defined by:

$$\left\{\begin{array}{l} H(x,\chi,A) \;:=\; \displaystyle\sum_{S\subset N}\left(\prod_{j\in S}\chi_j^2\|x_j\|^2\right)\mathbf{I} \\ + \displaystyle\sum_{R,S\subset N}\sum_{i\in R\cap S}\Big(\gamma_R(\chi)\gamma_S(\chi)A_R(x_{-i})A_S(x_{-i})^\star + \gamma_{R\backslash i}(\chi)\gamma_{S\backslash i}(\chi)A_R(x)\otimes A_S(x)\Big) \end{array}\right.$$

Then the regulation map is defined by

$$\left\{\begin{array}{l} R_M(x,\chi,A) \;:=\; H(x,\chi,A)^{-1} \\ \left(\displaystyle\sum_{S\subset N}\left(\alpha_S(A)(x) + \sum_{i\in S}\Big(\gamma_S(\chi)A_S(x_{-i}, c_i(x)) + \gamma_{S\backslash i}(\chi)\kappa_i(\chi)A_S(x)\Big)\right) - T_M(h(x,\chi,A))\right) \end{array}\right.$$

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